Holes in the static Einstein universe and the model of the cosmological voids

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Spherically symmetric, static model of the cosmological voids is constructed in the framework of the Tolman-Oppenheimer-Volkov equation with the cosmological constant. Extension of the Tooper result (dimensionless form of the TOV equation) is provided for non-zero Λ. Then, the equation is simplified in $\alpha \to 0$, $\lambda \to 0$, $\lambda/\alpha = const$ regime, suitable for largest structures in Λ-dominated universe. Voids are treated as an underdensity regions in the static Einstein universe. Both overdensity and underdensity (relative to static universe) solutions exist. They are identified with standard astrophysical spherical objects and voids, respectively. Model is tested against observed properties (the radius - the central density relation) and density profiles of voids. Analytical formulae for radial density contrast profile and radii of the voids are derived. Some consequences for cosmological n-body simulations are suggested. Hints on the dark matter/dark energy EOS filling the voids are provided.

I. INTRODUCTION

In both Newton's and Einstein's theories of gravitation a large set of static[1] astrophysical objects well approximated in spherical symmetry exist: moons, planets, brown dwarfs, stars, globular clusters, galactic clusters, and so on. Spherical shape of these objects is the most obvious and the most common effect of the universal attractive force. However, existence of roughly spherically symmetric [16] underdensity regions in the large-scale matter distribution in the universe, named voids [24], was an unexpected discovery [15, 31]. We will show, in the framework of the non-linearly perturbed static Einstein universe (thereafter s.E.u.), the existence of static, spherically symmetric objects devoid of matter relative to the surrounding. They belong to the same family of solutions as normal overdensity objects ranging from planets to galactic clusters. Spherical underdensity regions ("voids") are however directly related to the existence of cosmological constant Λ. Due to very small numerical value of Λ, they exist only for the largest-scale structures of universe: the voids. For smaller objects, Λ provides negligible contribution. With $\Lambda = 0$ only overdensity static solutions do exist.

Despite extensive literature on the static, spherically symmetric solutions of Einstein equations (see e.g: [4, 18, 22] and references therein) this particular branch was not discussed in the literature known to author. Papers of [13] and [1] deal with similar model, but only the overdensity solutions were found and analyzed.

The article goals are:

1. to show the existence of the *static* solutions with *positive* (!) pressure gradient

2. to derive and analyze basic equations for the generalized polytropic fluid spheres with non-zero Λ; this generalize result of Tooper [29]

3. to emphasize existence (and explore properties) of the two classes of the solutions; static Einstein universe is a boundary between them

4. to show repulsive "gravitational force" acting on the test particles in the vicinity of the "hole"

5. to use the above solutions in the spherically symmetric static model of the cosmological voids

6. to test the model against observed density profiles and $\rho_c - D_{void}$ relation for the voids

Meanwhile, renewed interest in static Einstein universe resulted in a large number of recently published papers, see e.g. [2, 3, 5, 7, 9, 10, 12, 19, 20, 23, 26, 30] and references in these papers

Article is organized as follows. In Sect. II we remind well known properties of the s.E.u, whose physical properties (density, pressure and radius) are used in next section, Sect. III, to simplify notation. In the Sect. III we analyze family of solutions to the Tolman-Oppenheimer-Volkov (see e.g. [27]) equation with the cosmological constant. Some surprising properties of the underdensity solutions are presented in Sect. IV. Generalization of the Tooper [29] result (valid only for polytropic EOS) is presented in Sect. V. Finally, in Sect. VI, attempt is made to fit solutions of the TOV-Λ (Tooper-λ) equation

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[1] More precisely, quasi-static, as all these objects evolve.

to the observed density profiles of the voids. Possible consequences of the results are outlined in the last section, Sect. VII.

II. STATIC EINSTEIN UNIVERSE

Traditionally, static universe is discussed in the context of the general Friedmann-Lemaitre-Robertson-Walker (FLRW) solution, see [6]. Robertson [25] put it into separate class "E".

For a brief introduction into s.E.u we restrict ourselves to the metric representing 3-sphere with radius $R(t)$:

$$ds^2 = -dt^2 + R(t)^2 \left[d\chi^2 + \sin^2 \chi \, d\Omega^2\right] \qquad (1)$$

Here, $d\Omega^2 = \sin^2 \theta d\phi^2 + d\theta^2$, and χ, θ, ϕ are spherical coordinates on unit 3-sphere. The Einstein equations are:

$$3\left(\frac{\dot{R}}{R}\right)^2 + \frac{3}{R^2} = 8\pi G\rho + \Lambda \qquad (2a)$$

$$3\frac{\ddot{R}}{R} + 4\pi G(\rho + 3p) = \Lambda \qquad (2b)$$

Additionally, we have to specify properties of matter filling the universe. We consider Equation Of State (thereafter EOS) of the barotropic ideal fluid:

$$p = p(\rho), \qquad (3)$$

where p - pressure, ρ - energy density.

Some of the examples widely used in cosmology are dust ($p = 0$) and relativistic gas ($p = \rho/3$). Later in the article we will use sub-class of barotropic EOS, *polytropic gas*:

$$p = K\rho^\gamma, \quad \gamma = 1 + \frac{1}{n}. \qquad (4)$$

If we assume radius of the 3-sphere constant in (2):

$$R(t) = R_E$$

then the Einstein's equations reduce to a simple system of algebraic equations:

$$8\pi G \, \rho_E = \frac{3}{R_E{}^2} - \Lambda \qquad (5a)$$

$$8\pi G \, p_E = -\frac{1}{R_E{}^2} + \Lambda \qquad (5b)$$

where EOS is given by (4).

Let us note, that linear combination of eqs. (5) gives (see also (2b)):

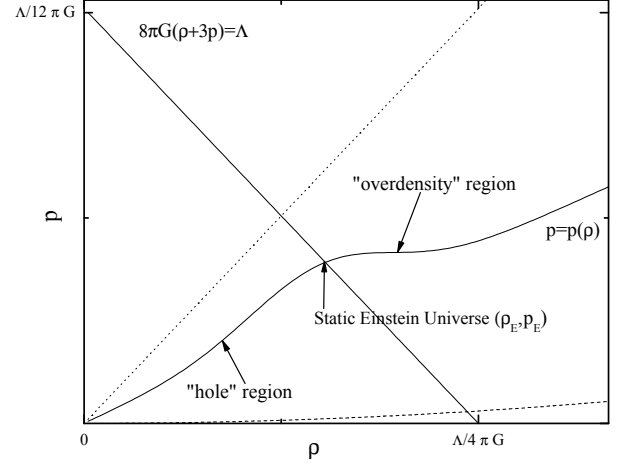$$4\pi G \, (\rho_E + 3\, p_E) = \Lambda. \qquad (6)$$

FIG. 1: Graphical solution of (6) with EOS (4). Dashed line represents eq. (6), and solid line corresponds to EOS. Dash-dotted line shows relativistic EOS $p = \rho/3$. Static 3-sphere solution (line crossing) exist for any "normal" EOS.

The gravitational field "source" $\rho + 3\, p$ required for static solution does not depend on the EOS. For all possible static Einstein universes is exactly the same. We will show later, that quantity $\rho_E + 3\, p_E$ is very important for inhomogeneous static solutions.

If we specify standard EOS[2] and the value of the cosmological constant Λ, then solution to (5) always exist and is unique (cf. Fig. 1). The well-known Einstein dust solution is:

$$\rho_E = \frac{\Lambda}{4\pi G}$$

with the radius:

$$R_E = \frac{1}{\sqrt{\Lambda}} \qquad (7)$$

In the case of relativistic EOS $p = \rho/3$ we have:

$$\rho_E = \frac{\Lambda}{8\pi G} \qquad (8)$$

$$R_E = \frac{\sqrt{3/2}}{\sqrt{\Lambda}} \qquad (9)$$

Therefore we can see, that size (R_E) and total mass of the Einstein static universe:

$$M_E = \rho_E \, V = 2\pi^2 R_E{}^3 \rho_E \qquad (10)$$

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[2] With $p, \rho > 0$ and $p \leq \rho/3$.

vary only slightly. Numerically:

$$1/\sqrt{\Lambda} < R_E < 1.22/\sqrt{\Lambda} \qquad (11)$$

and:

$$\frac{1.57}{G\sqrt{\Lambda}} > M_E > \frac{1.44}{G\sqrt{\Lambda}} \qquad (12)$$

If one allows more "exotic" matter (e.g. with negative pressure or density) it is possible to produce solutions spanning much more wide parameter range, see [11].

It is illustrative to compute R_E and ρ_E using the value of Λ recently determined from WMAP and SDSS observations. According to [28] cosmological constant is:

$$\frac{\Lambda c^2}{G} = 6.95 \times 10^{-27} kg/m^3$$

and density of the baryonic and dark matter is $\Omega_m + \Omega_b \simeq 0.41\,\Omega_\Lambda$. From (5) we get:

$$\frac{1}{2}\,\Omega_\Lambda < \Omega_{m+b}^{\text{s.E.u.}} < \Omega_\Lambda.$$

Radius of the static model (11) is proportional to:

$$\frac{1}{\sqrt{\Lambda}} \simeq 14.3 \text{ Gpc.}$$

S.E.u. however, once considered very seriously [8], is no longer considered a physical reality.

In the next section we would like to present alternative view of the static Einstein universe: one of the static spherically symmetric solutions of the TOV-Λ equation with barotropic fluid.

Our goal is to show the existence and explore properties of the wider than s.E.u. class of static solutions, namely spherically symmetric, possibly closed, universes with EOS (4). As we will show in the next sections, 3-sphere homogeneous solution is not an exclusive solution to the given problem. In particular, we discuss Tolman-Oppenheimer-Volkov equation with cosmological constant (next section), generalized Tooper equation with non-zero Λ (Sect. V) and additional "void equation" (28) in subsection VI A. Set of both numerical and analytical solutions to these equations is then used to fit observed properties of the cosmological voids.

III. TOV-Λ EQUATION

Using standard static spherically symmetric spacetime metric:

$$ds^2 = -e^{2\nu(r)}\,dt^2 + e^{2\lambda(r)}dr^2 + r^2\,d\Omega^2 \qquad (13)$$

we can easily derive Einstein field equations [27] for static ideal fluid with barotropic EOS (3). As these calculations

are extensively discussed in literature and textbooks we jump immediately to final form of the equations:

$$\frac{dp}{dr} = -G\,\frac{\left(\rho + \frac{p}{c^2}\right)\left(m + 4\pi r^3\,\frac{p}{c^2} - \frac{1}{3}\,\frac{r^3\Lambda c^2}{G}\right)}{r^2\left(1 - \frac{2\,Gm}{c^2 r} - \frac{1}{3}\,r^2\Lambda\right)} \qquad (14a)$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \qquad (14b)$$

Eq. (14a) is referred to as TOV-Λ equation. If we put $\Lambda = 0$ in (14a) we get standard TOV equation; if additionally we neglect pressure as a source of gravitational attraction and drop term $1 - 2m/r$ in the denominator, responsible for spacetime deformation in the presence of strong gravitational field, we get standard condition of hydrostatic equilibrium in Newtonian theory.

If we solve system (14) then metric (13) components are easily obtained from formulae:

$$e^{2\lambda} = \frac{1}{1 - 2\,m/r - 1/3\,r^2\Lambda} \qquad (15)$$

$$e^{2\nu} = -\int \frac{dp}{\rho + p} \equiv h(r) \qquad (16)$$

where h is the enthalpy.

As an exercise, we would like to look for constant solutions of the TOV-Λ equation. Let us substitute values $p(r) = p_E$ and $\rho(r) = \rho_E$ into (14). We get immediately condition (6). The g_{rr} component of the metric can be put in the form:

$$e^{2\lambda} = \frac{1}{1 - r^2/R_E^2}$$

with:

$$\frac{1}{R_E^2} = \Lambda - 8\pi G p_E.$$

We have got 3-sphere with radius R_E equal to value determined for static Einstein universe (5b), as expected.

Very interesting question arise: what happen if we take an initial value[3] for eq. 14a :

$$\rho(r = 0) \equiv \rho_0 \neq \rho_E$$

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The problem is solvable numerically. However, removable singularity is present at $r = 0$. To start numerical integration of (14) we follow standard procedure, replacing unknown functions with a series expansion:

$$\rho(r) \simeq \rho_0 + \alpha\,r + \frac{1}{2}\beta\,r^2 + \dots \qquad (17a)$$

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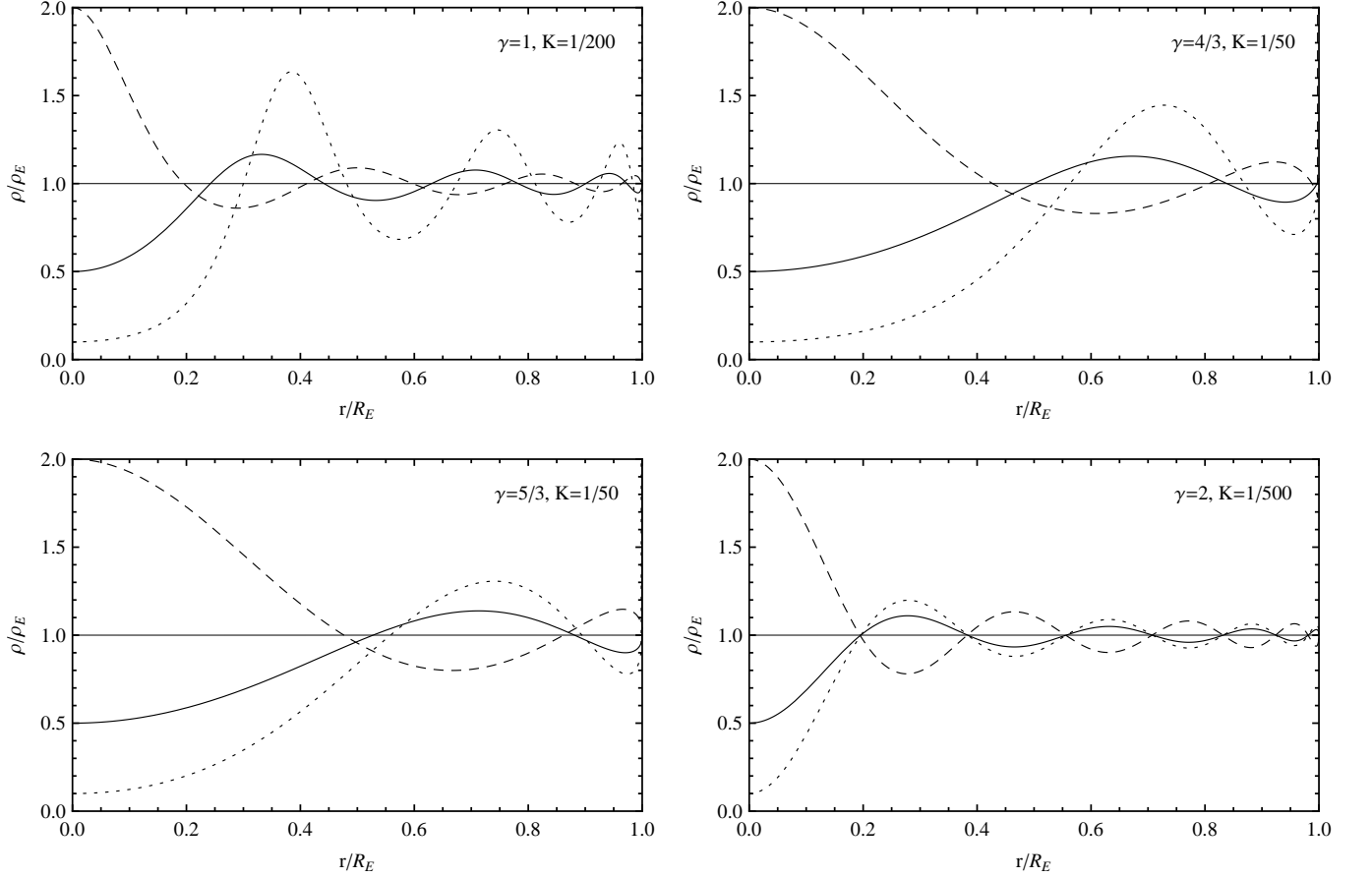[3] Initial value for second equation (14b) is $m(r=0) = 0$.

FIG. 2: Some selected numerical solutions to the (14) with barotropic EOS (4). Values used to obtain figures are: $\Lambda = 4\pi$, $G = 1$; respective density and radius of the dust s.E.u are $R_E = 1/\sqrt{4\pi}$ and $\rho_E = 1$. Initial values are: $\rho_0 = \rho_E/10$ (dotted) and $\rho_E/2$ (solid) for a "hole" and $2\rho_E$ for "overdensity region" (dashed).

$$p(r) \simeq p_0 + a\,r + \frac{1}{2}b\,r^2 + \dots \qquad (17b)$$

$$m(r) \simeq m_0 + A\,r + \frac{1}{2}B\,r^2 + \frac{1}{6}C\,r^3 + \dots. \qquad (17c)$$

Substituting series expansion into (14) immediately gives:

$$\rho(r) \simeq \rho_0 + \frac{1}{2}\beta r^2 + \dots \qquad (18a)$$

$$p(r) \simeq p_0 + \frac{1}{2}b r^2 + \dots \qquad (18b)$$

$$m(r) \simeq \frac{4}{3}\pi r^3 \rho_0 + \dots. \qquad (18c)$$

where:

$$b = \frac{4}{3}\pi G(p_0 + \rho_0)(\rho_E + 3p_E - \rho_0 - 3p_0) \qquad (19a)$$

and:

$$\beta = b \Big/ \left(\frac{\partial p}{\partial \rho}\right)_{\rho=\rho_0}. \qquad (19b)$$

From (19) we can distinguish three classes of the solutions:

(A) If $\rho_E + 3p_E < \rho_0 + 3p_0$ then $b < 0$ and we get solution similar to normal "star" (cluster, etc.) type solutions where density and pressure decreases outwards.

(B) $\rho_E + 3p_E > \rho_0 + 3p_0$ implies $b > 0$ i.e. both pressure and density grow with radius. This solutions describes "hole" in the background matter.

Finally, if $\rho_E + 3p_E = \rho_0 + 3p_0$ we get static Einstein universe, boundary solution between "clusters" and "voids".

Because class (B) of the solutions remained unnoticed[4]

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[4] In dimensionless form, eq. (22), it is much easier to see the

(to my knowledge), comment is required. The TOV-Λ equation usually is viewed as a perturbation to normal TOV equation. It is applied to the objects like stars. Typical initial value of density at r=0 exceeds that of the s.E.u. by many of orders of magnitude. Moreover, the integration is finished at point where $p = \rho = 0$. Therefore, no asymtotic density behavior is observed. Additionally, asymptotic behavior depends on polytropic, like for Lane-Emden equation. For fractional n solution cannot be extendended beyond $\rho < 0$. If the initial value of source density ($\rho + 3 p$) is however comparable to $\rho_E + 3 p_E$ (but still in the regime (A)), then the new behavior is observed. Instead of going to $\rho = 0$, density approaches constant value, exhibiting oscillatory behavior, cf. Fig. 2. It happens for relatively small subset of all possible initial conditions.

Surprisingly, starting the integration of (14) with central density in the regime (B), i.e. less than $\rho_E + 3 p_E$, we get qualitatively different solution. We get family of static, underdensity objects, again approaching constant density (Fig. 2). Analysis of the original TOV-Λ with general EOS (3) is very difficult task. Therefore, we have concentrated on its dimensionless (22) and simplified (28) forms. They are valid for polytropic EOS (4).

We will show in the Section IV that the existence of these objects is a manifestation of the repulsive gravity related to the non-zero value of the cosmological constant.

IV. PROPERTIES OF THE VOID MODEL IN THE FORM OF THE "HOLE"

"Holes" (Fig. 2) are very exotic objects. Density *increases* outwards and pressure gradient is *positive*. Newtonian intuition tells us, that matter inside given radius should attract particle outside. It is however simple to show, that test particle will be repelled from the hole[5] using relativistic approach.

Because our solution is spherically symmetric we may use the effective potential method:

$$V_{eff} = \left(1 - \frac{2m}{r} - \frac{1}{3}\Lambda r^2\right)\left(1 + \frac{L^2}{r^2} - \frac{E^2}{e^{2\nu}}\right) \quad (20)$$

where $m(r)$ and $\nu(r)$ are computed from (14b) and (16) using numerical solution (Fig. 3). If no local extremum exist, then circular orbits (both stable and unstable) are impossible and "force" is repulsive. This is the case of the hole (cf. Fig. 3, right panel). Therefore, we have shown,

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existence of the two solution classes, cf. Sect. V.

[5] Or attracted by the universe outside. Unfortunately, for the closed universe there is no clear method to define which part of the 3-sphere split by 2-sphere into two regions, is "outside" or "inside" 2-sphere. We remind, that this solution is not asymptotically flat, and describe spatially finite space

that the hole repels test particles[6] from inside. Without positive pressure gradient it cannot be static. This also explains why non-trivial EOS, e.g. (4), is required for this type of the solution. Dust particles will be repelled from the hole. Static Einstein universe is an exception, because of its high symmetry.

V. GENERALIZED TOOPER FLUID SPHERES AND DIMENSIONLESS FORM OF THE TOV-Λ EQUATION

By generalization of the Tooper [29] result, we can put TOV-Λ equations (14) in dimensionless form. This is possible because power-law equation of state (4) has been used. For general barotropic EOS, eq. (14) must be used instead.

Let us introduce new dimensionless radial variable:

$$x = A\,r,$$

and new functions:

$$\rho(r) = \rho_c\,\theta(x)^n, \quad (21a)$$

$$p(r) = p_c\,\theta(x)^{n+1}, \quad (21b)$$

$$m(r) = \frac{4\pi\rho_c}{A^3}v(x). \quad (21c)$$

Here, ρ_c is the density at the center ($r = 0$) and p_c is the central pressure.

Now, Eqns. (14) take the form:

$$\theta' + \frac{(1 + \alpha\theta)(v + \alpha x^3\,\theta^{n+1} - \lambda\,x^3/\alpha)}{x^2 - 2\alpha\,x\,v - \lambda\,x^4} = 0 \quad (22a)$$

$$v' = x^2\,\theta^n \quad (22b)$$

where:

$$A^2 = \frac{4\pi G\rho_c}{\alpha c^2}, \quad \lambda = \frac{\Lambda}{3\,A^2}, \quad \alpha = \frac{p_c}{\rho_c c^2}, \quad \gamma = 1 + \frac{1}{n}.$$

Initial conditions for system (22) are: $\theta(0) = 1$ and $v(0) = 0$. If cosmological constant (thus parameter λ) is equal to zero, and we neglect relativity parameter α, system (22) reduces to the Lane-Emden equation. However, if $\lambda \neq 0$, there is no simple Newtonian limit. The last term of the numerator in eq. (22a) diverge. We will use this property in subsection VI A to build the simple void model.

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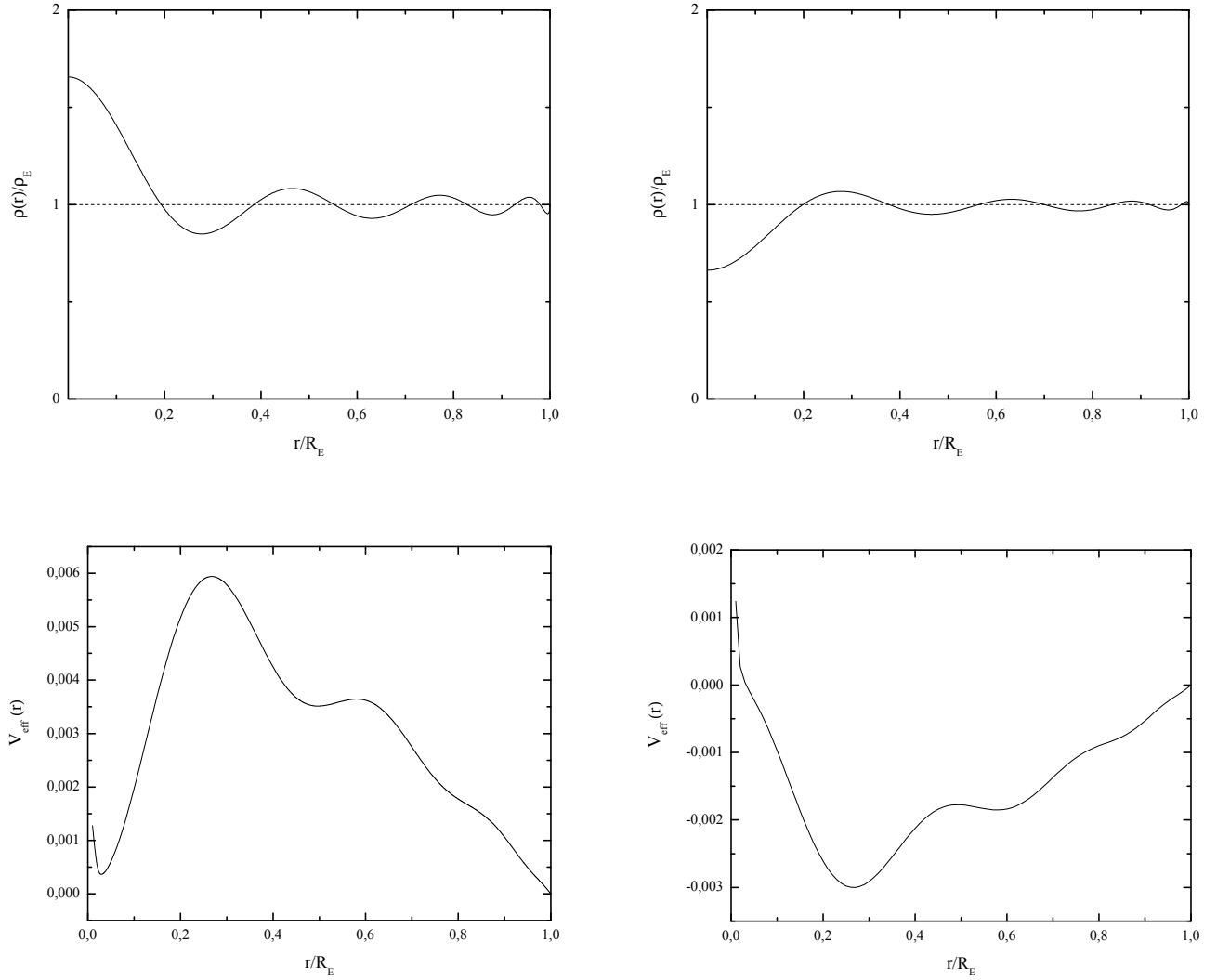[6] Thus, intriguing question arise about "force" between two holes.

FIG. 3: Density profile for overdensity object (upper left) and hole (upper right) with effective potential (20) with very small L below. Stable circular orbit exist very close to the center of overdensity region (lower left), while extremum do not exist inside a hole for identical L and E. Stable orbit, however, exist very close to the "surface" of the hole. This behavior is easy to understand if we note, that source of gravitational attraction/repulsion is over/under–density relative to values for s.E.u., not entire matter inside a given radius.

Behavior of the solution around $x = 0$ can be explored using series expansion. Repeating procedure used previously for (14), we get:

$$\theta(x) \simeq 1 + \frac{1+\alpha}{2}\left(\lambda/\alpha - \frac{1}{3} - \alpha\right) x^2 + \dots \qquad (23a)$$

$$v(x) = \frac{1}{3}\, x^3 + \dots \qquad (23b)$$

Let us note that constant solution $\theta(x) = 1$ (static Einstein universe) appears only if:

$$\lambda = \alpha(\alpha + 1/3) \equiv \lambda_c. \qquad (24)$$

Using (24) we rewrite (23a) in the form:

$$\theta(x) \simeq 1 + \frac{1+\alpha}{2\alpha}\left(\lambda - \lambda_c\right) x^2 + \dots. \qquad (25)$$

Eq. 25 shows that $\theta(x)$ decreases only if $\lambda < \lambda_c$; $\lambda = 0$ in particular.

The transformation from eqns. (14) to (22) clarify the number of the independent free parameters in (14). Density profile can be obtained from three-parameter special function $\theta_n(x; \alpha, \lambda)$ using (21a).

VI. FITTING "HOLE" MODEL TO OBSERVED PROFILES OF THE COSMOLOGICAL VOIDS

Radial density profile of the voids can be determined from astronomical observations. We have used data from Fig. 4 of [14] to compare static model with observations.

In general, static model based on Tooper-Λ equation (22) provide 4-parameter family of solutions. If EOS is given completely, i.e. without any of the free parameters[7], then solution is determined by just two parameters: central density and Λ. Solutions are enumerated by the central density ρ_0 and value of the cosmological constant Λ. In principle, value of Λ is known from e.g. Λ-CDM model. However, real voids are not placed in the static Einstein universe, but in Λ-CDM Universe. Some effective $\Lambda_{eff} \neq \Lambda_{CDM}$ could be used instead. Therefore we consider λ in (22) a free parameter. Radial profile of the void in [14] is normalized arbitrarily in radius. Therefore additional free parameter A must be used to scale solution of the TOV-Λ equation. Polytropic EOS (4) has been used in fitting with additional parameter, $n = 1/(\gamma - 1)$.

The most general fit:

$$\rho(r) = \rho_c\, \theta_n(Ar;\, \alpha, \lambda) \qquad (26)$$

includes five free parameters: central density of the void, ρ_c, constant n coming from polytropic EOS (Eq. 4), the relativity parameter α, the cosmological constant related parameter λ and one more parameter, A, used to scale solution of the TOV-Λ equation.

Representative fits are shown in Fig. 4. Overall quality of approximation is good. Four cases are presented. Because it is unclear why the data points are missing in Fig. 4 of [14] for $r < 0.2 R_{void}$, we consider two scenarios. In first, we use fixed central density. The value is determined by extrapolation of data from $0.25\, R_{void} < r < 0.85\, R_{void}$ to $r = 0$. Linear fit gives:

$$\rho(r) = pr + q$$

where:

$$p = 0.009 \pm 0.004, \quad q = 0.023 \pm 0.003 \quad (1\sigma).$$

This is consistent ($p \in (-0.0023, 0.021)$, at 95% confidence level) with constant density ($p = 0$) for $r < 0.85 R_{void}$. Assuming constant density we get:

$$\rho_c = 0.028 \pm 0.001 \quad (1\sigma),$$

and this value has been used in fits.

Four typical cases are presented in Fig. 4. Upper-right panel of Fig. 4 will be examined in greater detail in subsection VI A. All cases are summarized in Table I. Except

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[7] For example, if $p = \rho/3$ then $\alpha = 1/3$ and $n = \infty$.

TABLE I: Parameters used in Fig. 4.

ρ_c	n	$\log_{10}\alpha$	$\log_{10}\lambda$	A
0.028 (fixed)	2 (fixed)	-66.38	-65.86	1 (fixed)
0.028 (fixed)	2 (fixed)	-1.3	-0.57	0.65
0.028 (fixed)	8.6	0.66	2.6	0.027
0.004 ± 0.015	1.26 ± 0.29	-2.5 ± 1.5	-1.0 ± 0.2	1 (fixed)

for lower-left figure in Fig. 4, $\alpha \ll 1$ and $\lambda \ll 1$. Little can be told on the polytropic index n. For all four cases $1 < n < 9$. The best fit has been obtained with fixed central density (lower-left panel in Fig. 4). Value of λ is very large in this case, and polytropic index $n > 8$. This suggest that voids are Λ-dominated objects filled with ideal gas with a lot ($2n > 16$) of the degrees of freedom. This, unfortunately, contradicts results of subsection VI B. However, used density profile is the averaged one. Profiles for individual voids would work better, but there is no method to measure them. Too many degrees of freedom in model (26) combined with limited amount of data allow us to provide hints rather than constraints on EOS.

A. The "void equation"

The case with fixed polytropic index, $\gamma = 2$ and fixed central density (Fig. 4, upper-left) is particularly intriguing. Solution is not scaled in radius, i.e $A = 1$. We have just two free parameters: α and λ. Best fit parameters (upper-left panel in Fig. 4) are very small, and are poorly determined. Actually, from least-square fits we get random pairs of extremely small values. However, ratio λ/α is almost constant. In this situation we can simplify eqns. (22) a lot. Assuming that $\alpha \to 0$ and $\lambda \to 0$ and $\lambda/\alpha \to const$ we get:

$$\theta' + \frac{v - px^3}{x^2} = 0 \qquad (27a)$$

$$v' = x^2 \theta^n \qquad (27b)$$

where we denoted $p = \lambda/\alpha$. For $n = 1$ (initial conditions the same like for (22)) we get analytical solution for the density profile of the void:

$$\rho(r) = \rho_c\, \frac{\sin x - 3p \sin x + 3px}{x}, \quad x = Ar.$$

For $n \neq 1$ we still have to use numerics (see however eq. 31 and Fig. 5), but the basic equation is much simpler (compared to (22)):

$$\vartheta'' + \frac{2}{x}\vartheta' + \vartheta^n = 3p \equiv q^n, \quad \vartheta(0) = 1, \vartheta'(0) = 0. \quad (28)$$

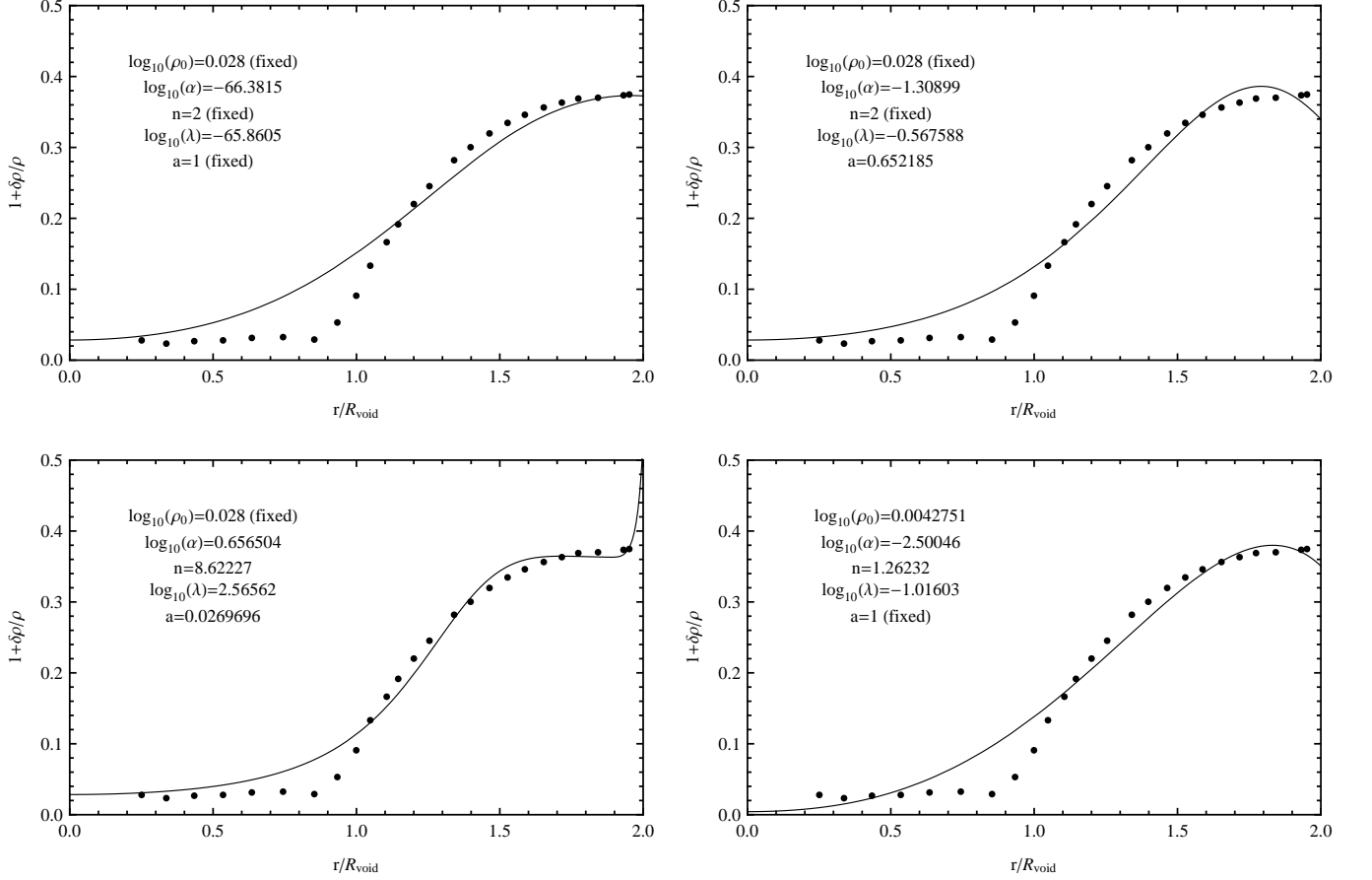For $q = 0$ eq. (28) reduces to the Lane-Emden equa-

FIG. 4: Best fit solution of the Tooper-Λ (26) and radial profile of void provided by [14]. Points show data extracted from "SGP" dataset.

tion. For $q \neq 0$ behavior of the solutions[8] for both $x \to 0$ and $x \to \infty$ changes dramatically. Here we restrict to positive solutions only: $\vartheta > 0$, $q > 0$ and $n > 0$.

For $x \ll 1$ we have:

$$\vartheta(x) \simeq 1 + \frac{q^n - 1}{6} x^2 + \dots,$$

and depending on sign of $q^n - 1$ we have constant, increasing or decreasing solution. For $x \to \infty$ we have obtained asymptotic expression:

$$\vartheta(x) \sim q - \frac{a \, \sin \omega x}{x}, \tag{29}$$

where:

$$\omega^2 = n \, q^{n-1}. \tag{30}$$

The constant a can be determined from numerical solution. Expression (29) provides surprisingly good (see

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[8] Scaling symmetry of the Lane-Emden equation $\zeta^s \vartheta(x/\zeta)$, where $s = 2/(1-n)$ is no longer valid if $q \neq 0$ in (28).

Fig. 5) overall fit to the solution in the entire interval $[0, \infty]$. Value of the constant a in (29) can be roughly determined from initial condition $\vartheta(0) = 1$. This gives $a = (q-1)/\omega$:

$$\vartheta(x) \simeq q - (q-1) \frac{\sin \omega x}{\omega x}. \tag{31}$$

Expression (31) can be used instead of numerical solution for (28) if large accuracy is not required. Maximum relative error is never larger than 15% for $t > 0$, and is below 7% for $1 < n < 5$ and $0 < q < 2$.

The most important in our application (a void model) is the leading constant in (29). Knowledge of the behavior at infinity allow us (in contrast to eqns. (14, 22)) to introduce the *density contrast*:

$$\frac{\delta \rho}{\rho} \equiv \frac{\rho - \rho_\infty}{\rho_\infty}. \tag{32}$$

Using:

$$\rho(r) = \rho_c \, \vartheta(Ax)^n, \quad \rho_\infty = \rho_c \, q^n$$

we have:

$$1 + \frac{\delta \rho}{\rho} = \left(\frac{\vartheta(Ax)}{q} \right)^n \tag{33}$$
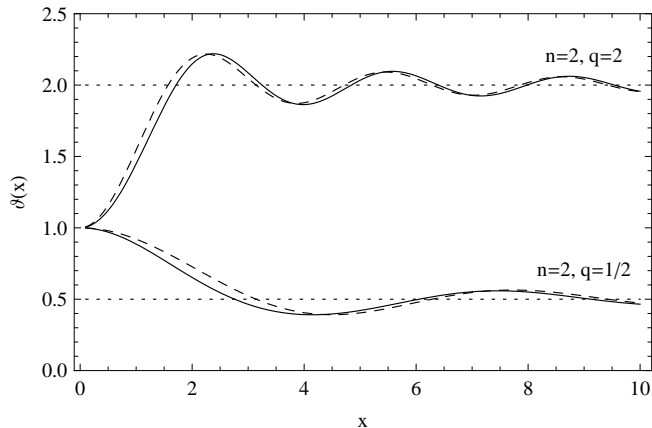
FIG. 5: Example solutions to the "void equation" (28) (solid line) and approximate analytical formula (31) (dashed line). Dotted horizontal lines show asymptotic value $\lim_{x\to\infty}\vartheta_n(x)=q$.
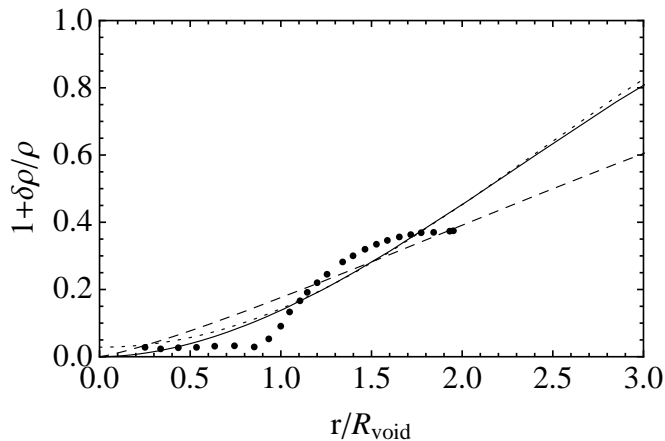

FIG. 6: Fitting results obtained with aid of the analytical formula (33, 31) for the density contrast in the void. See the text for the details.

where ϑ is solution to the eq. (28) or its approximation (31).

We have obtained void model much simpler and convenient compared to original function (26) which must be evaluated numerically. Using (33) and (31) we can fit density contrast using three parameters. Two constants: A and ω always appear as a product $A\omega$, and should be treated like a single parameter.

Typical fitting results are shown in Fig. 6. Quality is apparently lower compared to the Tooper-Λ fits (Fig. 4). But curves in Fig. 6 represent functions with well-defined and correct asymptotic behavior. Probably, the polytropic EOS (4) is too simple to reproduce fine structure of the density profile. Obtained values of the polytropic index n also represent some average value.

Three curves in Fig. 6 are presented. Solid line shows unconstrained fit. Obtained values are: $n = 0.9 \pm 0.2$,

$A\omega = 0.86\pm0.09$ and $q \to \infty$. Because $\rho_c \propto 1/q^n$ we have obtained void model that is empty at the very center. Similar result ($n = 3/5$, $A\omega = 0.58\pm0.02$ and $q \to \infty$) has been obtained (Fig. 6, dashed line) with fixed polytropic index of $n = 3/5$. If we fix the central density:

$$1/q^n = \rho_c/\rho_\infty$$

then $n = 1.05 \pm 0.2$ and $A\omega = 0.89 \pm 0.09$, see Fig. 6, dotted line.

This simplest model do not produce impressive results, and little information on the matter filling the void can be extracted. Unfortunately, data is missing in crucial region $r < 0.2R_{void}$, as well as for $r > 2R_{void}$. Extrapolating data we expect $\rho_0 \simeq 0.028$ and $\rho_0 \to 1$ for $r \gg R_{void}$. Sudden change of derivative at $r \simeq R_{void}$ strongly suggest different EOS, more complicated than (4).

B. Radius-central density relation for the voids

In addition to the void density contrast profile, $D_{void}-\rho_c$ relation has been published in [21]. We can use (31) to derive this relation. We define void radius as a point where density contrast becomes equal to 1 for the first time. Therefore, problem reduces simply to the first zero of the $\sin\omega x/(\omega x)$, i.e:

$$A\omega\,R_{void} = \pi.$$

Using ω from (30) and $1/q^n = \delta\rho_c$, we get:

$$R_{void} = \frac{\pi}{\sqrt{nA}}\,\delta\rho_c^{1-\gamma/2} = \frac{\pi}{\sqrt{nA}}\,\delta\rho_c^{\frac{n-1}{2n}}. \quad (34)$$

Here, $\delta\rho_c$ is the central density contrast plus 1. Using A and $\gamma(=1+1/n)$ as a free fit parameters, we get:

$$A = 0.23\pm0.01, \quad \gamma = 2.66\pm0.07 \quad \text{i.e.} \quad n = 0.602\pm0.025$$

According to our model, radius of the void can be approximated (see Fig. 7) using:

$$R_{void} = \frac{R_0}{\delta\rho_c^r} \quad [h^{-1}Mpc]$$

with:

$$R_0 = 17.6 \pm 0.6, \quad r = 0.33 \pm 0.03 \simeq \frac{1}{3}.$$

Polytropic exponent as large as $\gamma = 8/3$ suggest that strongly interacting matter (particles?) fill the voids. Ideal gas ($\gamma < 5/3$) is excluded in this model, because void radius would increase with central density.

VII. CONCLUSIONS

We have presented analysis of the spherically symmetric solutions of the TOV-Λ equation (14). We focus on
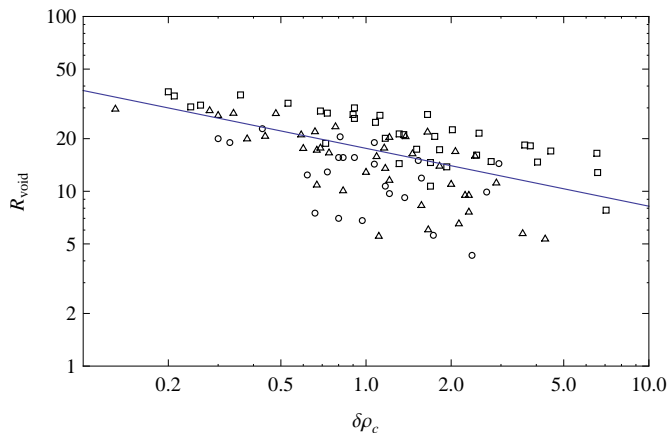
FIG. 7: Measured radii and central densities of the voids (Fig. 8 and Tables 2a-c of Ref. Lindner et al. [21]) fitted to our model equation (34). Fit is consistent with $n = 3/5$ i.e. $\gamma = 8/3$.

solution branch representing underdensity regions in the static Einstein universe. These solutions exist only if initial value obey $\rho_0 + 3p_0 < \rho_E + 3p_E$, where index E refer to values for static 3-sphere solution. It is shown that underdensity void-like *static* regions (in the presence of non-zero Λ and non-trivial EOS) are possible in general relativity. It contradicts common among cosmologist belief, that voids can exist only because of the expansion. In principle, they could exist even in the static universe as well.

Static solution with density increasing outwards is the most striking manifestation of the repulsive gravitational interaction in the presence of the cosmological constant. These solutions remain notoriously unnoticed and ignored, to my knowledge. For example, authors of the paper [1] seem excited with existence of the static generalization of the Einstein solution. But in their figures 4-6 pressure (solid lines) always decrease. This is not true for all of the solutions, cf. Fig. 2. We also point out that for closed static universe there is no such thing as "outward"("inward") direction. In general, however, solutions of (22) might not be regular for all values of n, α and λ. Singularity, if present, break this "two center" symmetry, and can be used to define "outward direction".

It is shown that hole repels test particles despite $\rho + 3p$ is positive inside. Instead, $\rho + 3p - \rho_E - 3p_E$ takes the role of gravity source, at least in this particular model. It can be either positive or negative. Therefore voids in static universe grow not only due to attraction of matter by structures outside or by expansion. They can increase because test particles are expelled from inside. This, in

turn, accelerate structure formation outside the void via "snow plough" mechanism of [17]. This behavior cannot be correctly approximated using standard N-body simulation. However, we can possibly mimic repulsive forces with small admixture of negative mass quasi-particles in N-body codes. These "particles" could represent underdensity "holes" (lack of the matter) similarly to holes in semiconductor physics, representing lack of the electron. Structure formation in the static Einstein universe is of purely academic interest. Similar effects in the real expanding universe are not excluded in the non-linear regime, however.

The static model has been applied directly to the observed profiles of voids. Despite the real universe is expanding, model works surprisingly well, cf. Fig. 4. Density profile can be fitted even with the simple polytropic equation of state in the framework of the Tooper-Λ equation (22). This might be interpreted twofold: (1) physically motivated toy-model which works even if the real object do not satisfy all assumptions (2) manifestation of the unknown properties of the dark matter or dark energy EOS filling the void and making it static.

To explore the latter possibility, we have derived simplified version of the eq. (22). This "void equation" (28) is well-suited for studying largest cosmological structures. It comes from a limit $\alpha \to 0$, $\lambda \to 0$ and $\lambda/\alpha = const$ in (22) that is neither Newtonian nor Λ-dominated. Basically, (28) is the Lane-Emden modified by the presence of the constant q^n on the right-hand-side. The constant $q \neq 0$ break scaling symmetry present in the Lane-Emden equation. Unexpectedly, solution for $q > 0$ is far less complicated than that of Lane-Emden equation. Asymptotic behavior (cf. (29)) is the same for all n's. Approximate analytical form of the solution (31) allow us to derive $D_{void} - \rho_c$ relation (34). Comparing with data from void catalogue [21] we get $\gamma \simeq 8/3$ in (4).

We have explored hypothesis, that the voids, are not completely empty. We have assumed that they are filled with unknown form of the "dark" matter. For simplicity, and due to poor knowledge of the dark matter/energy physics, polytropic EOS (4) has been used. Static model requires also that "voids" are already decoupled from Hubble flow, like e.g. galaxy clusters. By extrapolation of the Hoyle & Vogeley [14] result for radial density profile to $r = 0$, and from $D_{void} - \rho_c$ relation [21] we get polytropic index in the range $0.6 \lesssim n \lesssim 1.25$. This suggest non-ideal gas, possibly strongly interacting. On the contrary, if we use central density as a free parameter we obtain (i) nearly empty central region and (ii) much larger polytropic index $1 \lesssim n \lesssim 9$. This do not exclude ideal gas, but its nature, e.g. number of the degrees of freedom cannot be determined.

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